Name: Solutions Section: ______

Definitions

1. Define Span$\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3\}$.

$$= \left\{ \vec{b} \text{ in } \mathbb{R}^m \ : \ \vec{b} = c_1\cdot\vec{v}_1 + c_2\cdot\vec{v}_2 + c_3\cdot\vec{v}_3 \text{ for some } c_1, c_2, c_3 \text{ in } \mathbb{R} \right\}$$

= the set of all $\vec{b}$ that can be gotten by scaling & adding $\vec{v}_1, \vec{v}_2, \vec{v}_3$

2. Define linear **Independence** of vectors $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3\}$.

$\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly independent

$\Leftrightarrow$

the equation $x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 = \vec{0}$ has <u>ONLY</u> the trivial solution

3. Define "$T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation"

$T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$

<u>AND</u>

$T(c\cdot\vec{u}) = c\cdot T(\vec{u})$

for <u>ALL</u> $\vec{u}, \vec{v}$ in $\mathbb{R}^n$ and <u>ALL</u> c in $\mathbb{R}$.

4. Define "$T : \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one"

$T(\vec{x}) = \vec{b}$ has <u>AT MOST</u> one solution for each $\vec{b}$ in $\mathbb{R}^m$

5. Define "$T : \mathbb{R}^n \to \mathbb{R}^m$ is onto"

$T(\vec{x}) = \vec{b}$ has <u>AT LEAST</u> one solution for each $\vec{b}$ in $\mathbb{R}^m$

Name: ____________________ Section: ____________________

General Linear Transformations

1. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined so that $T(\vec{\mathbf{x}}) = A\vec{\mathbf{x}}$ where

$$A = \begin{bmatrix} 1 & -3 & -2 \\ -1 & 2 & 3 \\ 1 & -4 & -1 \end{bmatrix}, \qquad \vec{\mathbf{u}} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \qquad \vec{\mathbf{b}} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \qquad \text{and} \qquad \vec{\mathbf{c}} = \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix}$$

(a) Compute $T(\vec{\mathbf{u}})$.

$$T(\vec{u}) = \begin{bmatrix} 1 & -3 & -2 \\ -1 & 2 & 3 \\ 1 & -4 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \\ 2 \end{bmatrix}$$

(b) Find all solutions to the equation $T(\vec{\mathbf{x}}) = \vec{\mathbf{b}}$ solve $A\vec{x} = \vec{b}$

Solve
$$\left[\begin{array}{ccc|c} 1 & -3 & -2 & -1 \\ -1 & 2 & 3 & 2 \\ 1 & -4 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -3 & -2 & -1 \\ 0 & -1 & 1 & 1 \\ 0 & -1 & 1 & 1 \end{array}\right] \begin{array}{l} \\ r_2+r_1 \\ r_3-r_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & -3 & -2 & -1 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -5 & -4 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} r_1-3r_2 \\ \\ \\ \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 5 & -4 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right] \iff \begin{cases} x_1 \quad +5x_3 = -4 \\ \quad x_2 - x_3 = -1 \\ x_3 \text{ free} \end{cases} \iff \begin{cases} x_1 = -4-5x_3 \\ x_2 = -1+x_3 \\ x_3 \text{ free} \end{cases}$$

(c) Is $\vec{\mathbf{c}}$ in the range of T? Justify your answer. solve $A\vec{x} = \vec{c}$

$$\left[\begin{array}{ccc|c} 1 & -3 & -2 & -1 \\ -1 & 2 & 3 & 2 \\ 1 & -4 & -1 & 5 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -3 & -2 & -1 \\ 0 & -1 & 1 & 1 \\ 0 & -1 & 1 & 6 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -3 & -2 & -1 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 5 \end{array}\right] \leftarrow \text{system is inconsistent} \Rightarrow \underline{\text{NO}} \text{ solution.}$$

Name: ____________________ Section: ____________________

2. (No Computation) Define $T : \mathbb{R}^3 \to \mathbb{R}^2$ be defined so that $T(\vec{\mathbf{x}}) = A\vec{\mathbf{x}}$ where

$$A = \begin{bmatrix} 2 & 3 & -1 \\ 1 & -3 & 2 \end{bmatrix}$$

(a) What is the domain of T?

Domain = input space = $\mathbb{R}^3$

(b) What is the co-domain of T?

Co domain [redacted] = output space = $\mathbb{R}^2$

(c) Describe the Range of T as the span of a set of vectors.

$$\text{Range}(T) = \text{Span}\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \end{bmatrix} \right\}.$$

3. (No Computation) How many rows and columns must a matrix A have in order to define a mapping from $\mathbb{R}^5$ into $\mathbb{R}^7$ by the rule $T(\vec{\mathbf{x}}) = A\vec{\mathbf{x}}$?

inputs = columns = 5

outputs = rows = 7

$\boxed{\text{need a } 7 \times 5 \text{ matrix } A}$

Name: ______________________ Section: ______________________

4. Let $\vec{\mathbf{x}} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\vec{\mathbf{v}}_1 = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$, and $\vec{\mathbf{v}}_2 = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$, and let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation that maps $\vec{\mathbf{x}}$ to $x_1\vec{\mathbf{v}}_1 + x_2\vec{\mathbf{v}}_2$. Find a matrix A so that $T(\vec{\mathbf{x}}) = A\vec{\mathbf{x}}$ for every $\vec{\mathbf{x}}$.

$$T(\vec{x}) = x_1\vec{v}_1 + x_2\vec{v}_2$$
$$= x_1\begin{bmatrix} 2 \\ -5 \end{bmatrix} + x_2\begin{bmatrix} 3 \\ -2 \end{bmatrix}$$
$$= \underbrace{\begin{bmatrix} 2 & 3 \\ -5 & -2 \end{bmatrix}}_{A}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

5. Find the standard matrix of the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that sends $\vec{\mathbf{e}}_1$ to $\vec{\mathbf{e}}_1 - 3\vec{\mathbf{e}}_2$ and leaves $\vec{\mathbf{e}}_2$ unchanged.

$$T(\vec{e}_1) = \vec{e}_1 - 3\vec{e}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} - 3\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$
$$T(\vec{e}_2) = \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$A = \begin{bmatrix} T(e_1) & T(e_2) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$$

Name: ______________________ Section: ______________________

6. Prove that $T(\vec{\mathbf{x}}) = \underline{3\vec{\mathbf{x}}}$ is a linear transformation.

$P \leftrightarrow Q$ | $T(\vec{x})$ is linear $\iff$ $T(c\vec{u} + d\vec{v}) = c \cdot T(\vec{u}) + d\,T(\vec{v})$ for all $\vec{u}, \vec{v}$ and all c, d

Q | <u>compute:</u>

$$T(c\vec{u} + d\vec{v}) = 3 \cdot (c\vec{u} + d\vec{v})$$

$$= c \cdot \underline{\underline{3\vec{u}}} + d \cdot \underline{\underline{3\vec{v}}}$$

$$= c \cdot T(\vec{u}) + d \cdot T(\vec{v}) \checkmark$$

P | therefore T is linear

7. Prove that $T(\vec{\mathbf{x}}) = \boxed{3\vec{\mathbf{x}} + 1}$ is not linear transformation.

$P \leftrightarrow Q$ | $T(\vec{x})$ is linear $\iff$ $T(c\vec{u} + d\vec{v}) = c \cdot T(\vec{u}) + d \cdot T(\vec{v})$ for all $\vec{u}, \vec{v}$ & c, d

<u>compute</u>

$$T(c\vec{u} + d\vec{v}) = 3(c\vec{u} + d\vec{v}) + 1$$

$\neg Q$

$$= \underbrace{c \cdot 3\vec{u}}_{(\text{NOT } c\,T(\vec{u}))} + \underbrace{d \cdot 3\vec{v} + 1}_{(\text{NOT } d \cdot T(\vec{v}))}$$

[redacted]

$$\neq c \cdot T(\vec{u}) + d \cdot T(\vec{v})$$

$\neg P$ | therefore T is <u>NOT</u> linear

Name: ______________________ Section: ______________________

8. Let transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation with

$$T(\vec{\mathbf{e}}_1) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \text{ and } T(\vec{\mathbf{e}}_2) = \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}.$$

(a) Find the standard matrix A of T.

$$A = \begin{bmatrix} 1 & -1 \\ 0 & 3 \\ 1 & 0 \end{bmatrix}$$

(b) Determine if the transformation T is one-to-one.

T one-to-one $\iff$ columns of A are independent.

Count solutions to $A\vec{x} = \vec{0}$

$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 3 & 0 \\ 1 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 3 & 0 \\ 0 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$\uparrow$ $\uparrow$
pivot in each column
$\Rightarrow$
unique solution
$\Rightarrow$
A is indep

T is one-to-one.

(c) Determine if the transformation T is onto.

T is onto $\iff$ columns of A span $\mathbb{R}^3$

$\iff$ pivot in each row.

But no pivot in Row 3

So T is not onto.

Name: ______________________ Section: ______________________

9. Let transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation with

$$T(\vec{\mathbf{e}}_1) = \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix}, \; T(\vec{\mathbf{e}}_2) = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \text{ and } T(\vec{\mathbf{e}}_3) = \begin{bmatrix} 0 \\ 4 \\ 1 \end{bmatrix}.$$

(a) Find the standard matrix A of T.

$$A = \begin{bmatrix} 2 & 2 & 0 \\ -2 & 2 & 4 \\ 0 & 1 & 1 \end{bmatrix}$$

(b) Determine if the transformation T is one-to-one.

T is one-to-one $\Leftrightarrow$ columns of A are independent

$$\left[\begin{array}{ccc|c} 2 & 2 & 0 & 0 \\ -2 & 2 & 4 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 2 & 2 & 0 & 0 \\ 0 & 4 & 4 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 2 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 2 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\uparrow$

<u>no</u> pivot in col 3 $\Rightarrow$ A <u>not</u> indep $\Rightarrow$ T <u>NOT</u> one-to-one

(c) Determine if the transformation T is onto.

T is onto $\Leftrightarrow$ columns of A span $\mathbb{R}^3$

$\Leftrightarrow$ pivot in each row.

<u>But</u>

No pivot in Row 3

$\Rightarrow$ T <u>NOT</u> onto.

Name: ______________________ Section: ______________________

A is 3x2 $\begin{bmatrix} * & * \\ * & * \\ * & * \end{bmatrix}$

10. Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^3$ is a linear transformation with standard form matrix A.

    Prove that T is *not* onto. (Cite all relevant definitions and theorems by number).

$P \Leftrightarrow Q$ T is onto $\Leftrightarrow$ columns of A span $\mathbb{R}^3$

$\Leftrightarrow$ A has pivot in each row

<u>But</u>

$\neg Q$ you cannot fit 3 pivots into 2 columns

<u>So</u>

$\neg P$ T cannot be onto.

11. Give an example of a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ that *is* one-to-one (Hint: define T by choosing its standard matrix).

let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ then $T(\vec{x}) = A\vec{x}$ <u>is</u> one-to-one.

12. Suppose $T : \mathbb{R}^2 \to \mathbb{R}^3$ is linear. Is it true that T is one-to-one if and only if T is onto? Why doesn't this violate the invertible matrix theorem?

<u>NO.</u> But the invertible matrix theorem <u>ONLY</u> applies to square matrices.

Name: ______________________ Section: ______________________

A is 2x3 $\begin{bmatrix} * & * & * \\ * & * & * \end{bmatrix}$

13. Suppose $T : \mathbb{R}^3 \to \mathbb{R}^2$ is linear. Prove that T cannot be one-to-one.

$P \leftrightarrow Q$ | T is one-to-one $\Leftrightarrow$ the standard matrix A has independent columns

$\Leftrightarrow$ A has a pivot in each column.

$\neg Q$ | But you cannot fit 3 pivots into 2 rows

$\neg P$ | So T cannot be one-to-one

14. Give an example of a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ that *is* onto (Hint: define T by choosing its standard matrix).

let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

then $T(\vec{x}) = A\vec{x}$ is onto.

15. Suppose $T : \mathbb{R}^3 \to \mathbb{R}^2$ is linear. Is it true that T is one-to-one if and only if T is onto? Why doesn't this violate the invertible matrix theorem?

No.

Because the IMT ONLY applies to square matrices.

Name: ______________________ Section: ______________________

Matrix Operations

1. Compute the following matrix operations, or explain why they are undefined.

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 3 & -2 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 1 \\ 3 & -2 \\ 5 & 1 \end{bmatrix}$$

(a) AB

$$= \underset{2\times3}{\begin{bmatrix} 2 & 0 & 1 \\ 3 & -2 & 1 \end{bmatrix}} \underset{3\times2}{\begin{bmatrix} 0 & 1 \\ 3 & -2 \\ 5 & 1 \end{bmatrix}} = \underset{2\times2}{\begin{bmatrix} 5 & 3 \\ -1 & 8 \end{bmatrix}}$$

(b) BA

$$= \underset{3\times2}{\begin{bmatrix} 0 & 1 \\ 3 & -2 \\ 5 & 1 \end{bmatrix}} \underset{2\times3}{\begin{bmatrix} 2 & 0 & 1 \\ 3 & -2 & 1 \end{bmatrix}} = \underset{3\times3}{\begin{bmatrix} 3 & -2 & 1 \\ 0 & 4 & 1 \\ 13 & -2 & 6 \end{bmatrix}}$$

(c) A^T

$$= \begin{bmatrix} 2 & 0 & 1 \\ 3 & -2 & 1 \end{bmatrix}^T = \begin{bmatrix} 2 & 3 \\ 0 & -2 \\ 1 & 1 \end{bmatrix}$$

(d) $2A - 3B$

$$2\begin{bmatrix} 2 & 0 & 1 \\ 3 & -2 & 1 \end{bmatrix} - 3\begin{bmatrix} 0 & 1 \\ 3 & -2 \\ 5 & 1 \end{bmatrix}$$ DNE (dimensions don't match)

(e) $2A^T - 3B$

$$2\begin{bmatrix} 2 & 3 \\ 0 & -2 \\ 1 & 1 \end{bmatrix} - 3\begin{bmatrix} 0 & 1 \\ 3 & -2 \\ 5 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 0 & -4 \\ 2 & 2 \end{bmatrix} + \begin{bmatrix} 0 & -3 \\ -9 & 6 \\ -15 & -3 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ -9 & 2 \\ -13 & -1 \end{bmatrix}$$

Name: ______________________ Section: ______________________

2. Let

$$A = \begin{bmatrix} 1 & 5 & 0 \\ 1 & 1 & 1 \\ 1 & 3 & 0 \end{bmatrix}_{3\times 3}$$

(a) State the domain and co-domain of the transformation T defined by $T(\vec{\mathbf{x}}) = A\vec{\mathbf{x}}$.

domain = input space $= \mathbb{R}^3$

codomain [redacted] = output space $= \mathbb{R}^3$

(b) Compute A^{-1}.

$$\left[\begin{array}{ccc|ccc} 1 & 5 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 3 & 0 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 5 & 0 & 1 & 0 & 0 \\ 0 & -4 & 1 & -1 & 1 & 0 \\ 0 & -2 & 0 & -1 & 0 & 1 \end{array}\right] \begin{array}{l} \\ r_2 - r_1 \\ r_3 - r_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 5 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & -2 \\ 0 & -2 & 0 & -1 & 0 & 1 \end{array}\right] \begin{array}{l} \\ r_2 - 2r_3 \\ \\ \end{array} \sim \left[\begin{array}{ccc|ccc} 1 & 5 & 0 & 1 & 0 & 0 \\ 0 & -2 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & -2 \end{array}\right]$$

$1 - 5/2$

$2/2 - 5/2 = -3/2$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 5 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1/2 & 0 & -1/2 \\ 0 & 0 & 1 & 1 & 1 & -2 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -3/2 & 0 & 5/2 \\ 0 & 1 & 0 & 1/2 & 0 & -1/2 \\ 0 & 0 & 1 & 1 & 1 & -2 \end{array}\right] \begin{array}{l} r_1 - 5r_2 \\ \\ \\ \end{array}$$

$$A^{-1} = \begin{bmatrix} -3/2 & 0 & 5/2 \\ 1/2 & 0 & -1/2 \\ 1 & 1 & -2 \end{bmatrix}$$

(c) Let $\vec{\mathbf{b}} = \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix}$. Use the inverse of A to solve the matrix equation $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$.

$$A\vec{x} = \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix} \Rightarrow \vec{x} = A^{-1}\begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} -3/2 & 0 & 5/2 \\ 1/2 & 0 & -1/2 \\ 1 & 1 & -2 \end{bmatrix}\begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 0 \\ -4 \end{bmatrix}$$

Name: ______________________ Section: ______________________

3. Rewrite $(AB)^{-1}(B + A)$ using the properties of matrix operations.

   (Careful: Multiplication is *not* commutative).

$$(AB)^{-1}(B+A)$$

RIGHT distribute

$$= (AB)^{-1}B + (AB)^{-1}A$$

$$= B^{-1}A^{-1}\cdot B + B^{-1}\underline{A^{-1}A}$$

$$= B^{-1}A^{-1}B + B^{-1}I_n$$

$$= B^{-1}A^{-1}B + B^{-1}$$ ← cannot rewrite

4. Rewrite $(B + A)(AB)^{-1}$ using the properties of matrix operations.

   (Careful: Multiplication is *not* commutative).

$$(B+A)(AB)^{-1}$$

left distribute

$$= B(AB)^{-1} + A(AB)^{-1}$$

$$= B\cdot B^{-1}A^{-1} + AB^{-1}A^{-1}$$

$$= I_n\cdot A^{-1} + AB^{-1}A^{-1}$$

$$= A^{-1} + AB^{-1}A^{-1}$$ ← cannot rewrite

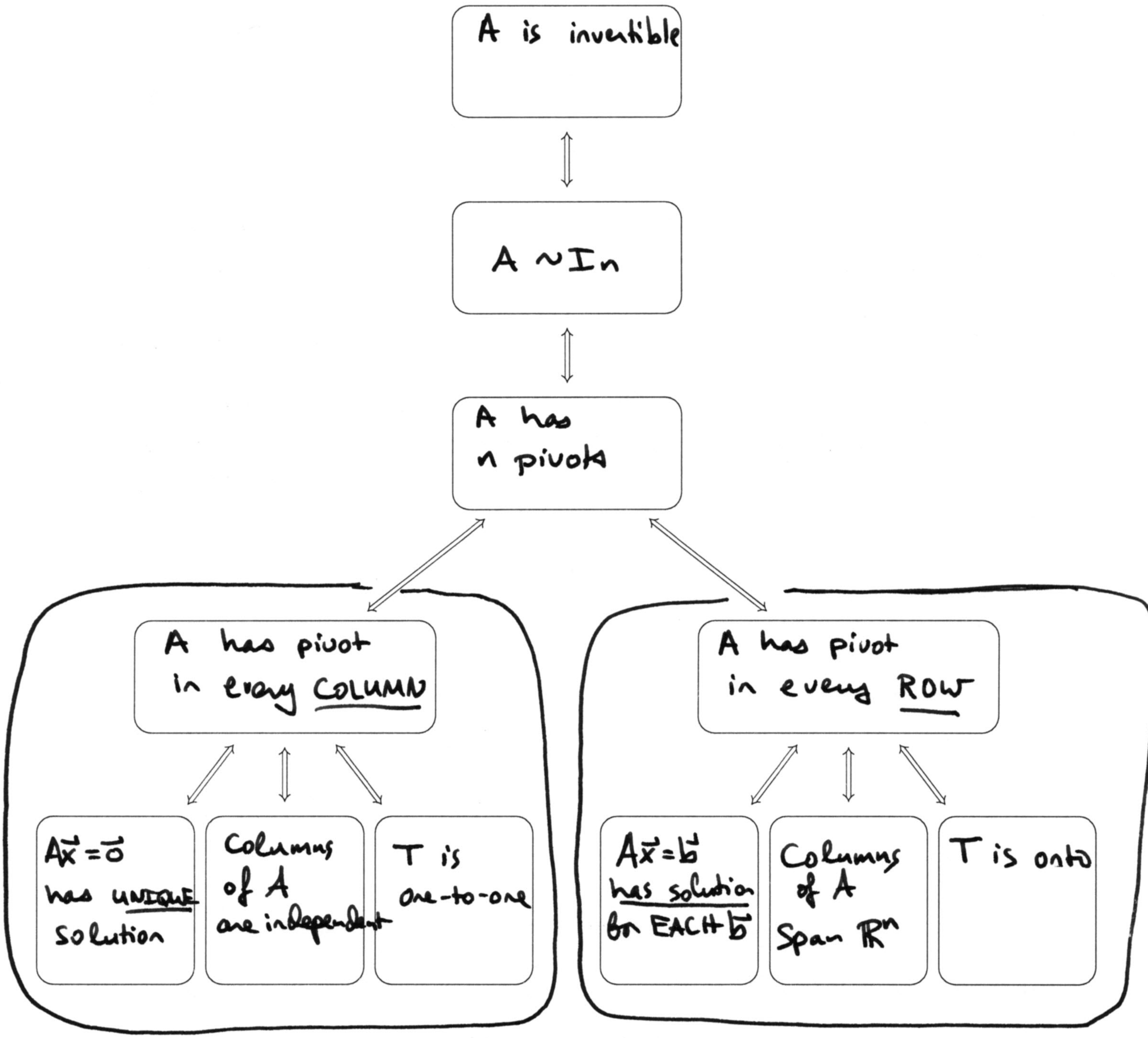
Name: ____________________ Section: ____________________

Square Matrices

The Invertible Matrix Theorem (IMT)

Suppose that A is an $n \times n$ matrix, and that $T : \mathbb{R}^n \to \mathbb{R}^n$ is defined by $T(\vec{\mathbf{x}}) = A\vec{\mathbf{x}}$.

Then, the following are equivalent.



Name: ____________________ Section: ____________________

1. Suppose that A is an $n \times n$ matrix. Prove the following statements by "walking through the tree" of the Invertible Matrix Theorem. **You must show every step**.

   (a) Suppose that an $n \times n$ matrix A is invertible.
   Prove that the columns of A span $\mathbb{R}^n$.

If A is invertible $\Rightarrow A \sim I_n$
$\Rightarrow$ A has n pivots
$\Rightarrow$ A has pivot in each Row
$\Rightarrow$ columns of A span $\mathbb{R}^n$

   (b) Suppose that an $n \times n$ matrix A is not invertible. Prove that the columns of A are linearly dependent.

If A is NOT invertible $\Rightarrow A \not\sim I_n$
$\Rightarrow$ A doesn't have n pivots
$\Rightarrow$ doesn't have pivot in each column
$\Rightarrow$ columns of A are dependent

   (c) Suppose that A is an $n \times n$ matrix, and that $A\vec{\mathbf{x}} = \vec{\mathbf{0}}$ has a unique solution. Prove that A is invertible.

If $A\vec{x} = \vec{0}$ has a unique solution
$\Rightarrow$ A has pivot in each column
$\Rightarrow$ A has n pivots
$\Rightarrow A \sim I_n$
$\Rightarrow$ A is invertible

   (d) Suppose that A is an $n \times n$ matrix, and that $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ does not have a solution for some $\vec{\mathbf{b}}$ in $\mathbb{R}^n$. Prove that A is not invertible.

If $A\vec{x} = \vec{b}$ doesn't always have a solution
$\Rightarrow$ A doesn't have a pivot in each row
$\Rightarrow$ A doesn't have n pivots
$\Rightarrow A \not\sim I_n$
$\Rightarrow$ A not invertible.

Name: ______________________ Section: ______________________

2. Determine if the following matrices are invertible using as few computations as possible.

square ✓ (a) $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

Note: Neither column is a multiple of other AND $n=2$

$\Rightarrow$ columns are independent

$\Rightarrow$ A is invertible

Recall

$(n=2) \Rightarrow \left(\text{Indep} \Leftrightarrow \text{Neither column is a multiple of the other} \right)$

square ✓ (b) $A = \begin{bmatrix} -2 & 6 \\ 1 & -3 \end{bmatrix}$

Note: $\begin{bmatrix} 6 \\ -3 \end{bmatrix} = (-3)\begin{bmatrix} -2 \\ 1 \end{bmatrix}$

and $n=2$

$\Rightarrow$ columns are Dependent

$\Rightarrow$ A is NOT invertible

square ✓ (c) $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$

NOTE: $A \sim I_3$

So A is invertible

Square ✓ (d) $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 6 \\ 0 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{bmatrix} \sim [$

↑ no pivot in 2nd column

$\Rightarrow$

Doesn't have 3 pivots

$\Rightarrow$

A is NOT invertible.

Name: ____________ Section: ____________

Theorems

Theorem 2 There are 3 cases for the reduced echelon form of a linear system's augmented matrix

NONZERO

1. The system has 0 solutions if it contains $[0 \cdots 0 \mid \blacksquare]$
2. The system has 1 solutions if it has pivot in each coeff. COLLUMN
3. The system has ∞-many solutions if some coeff. COLUMN lacks a pivot

Theorem 4: The columns of an $m \times n$ matrix A span $\mathbb{R}^m$

if and only if there is a pivot in every ROW.

Shortcuts to Recognize Dependence

- *If* one column of A is a multiple of another, *then* the columns of A are linearly dependent.
- *If* $\{\vec{\mathbf{a}}_1, \dots, \vec{\mathbf{a}}_n\}$ contains $\vec{\mathbf{0}}$, *then* $\{\vec{\mathbf{a}}_1, \dots, \vec{\mathbf{a}}_n\}$ is linearly dependent.
- *If* an $m \times n$ matrix A has more columns than rows (if $n > m$), *then* the columns of A are linearly dependent.

Theorem 5 If A is an $m \times n$ matrix, $\vec{\mathbf{u}}, \vec{\mathbf{v}} \in \mathbb{R}^n$ and $c \in \mathbb{R}$, *Then*

- $A(\vec{\mathbf{u}} + \vec{\mathbf{v}}) =$ $A\vec{u} + A\vec{v}$
- $A(c \cdot \vec{\mathbf{u}}) =$ $c \cdot A\vec{u}$

Properties of Linear Transformations

- If T is linear, then $T(\vec{\mathbf{0}}) =$ $\vec{0}$
- T is linear $\iff T(c \cdot \vec{\mathbf{u}} + d \cdot \vec{\mathbf{v}}) =$ $c \cdot T(\vec{u}) + d \cdot T(\vec{v})$

Theorem 10 Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be linear.

Then there is a unique $m \times n$ matrix A s.t. $T(\vec{\mathbf{x}}) = A\vec{\mathbf{x}}$.

In Fact, $A =$ $\begin{bmatrix} T(e_1) & \cdots & T(e_n) \end{bmatrix}$

Theorem 12 Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be linear with standard matrix A. T

(a) T is onto $\iff$ the columns of A span $\mathbb{R}^m$ $\iff$ A has pivot in each ROW

(b) T is one-to-one $\iff$ the columns of A are independent $\iff$ A has pivot in each COLUMN